

# THE AJ-CONJECTURE AND CABLED KNOTS OVER THE FIGURE EIGHT KNOT

DENNIS RUPPE

ABSTRACT. We show that most cabled knots over the figure eight knot in  $S^3$  satisfy the  $AJ$ -conjecture, in particular, any  $(r, s)$ -cabled knot over the figure eight knot satisfies the  $AJ$ -conjecture if  $r$  is not a number between  $-4s$  and  $4s$ .

## 1. INTRODUCTION

For a knot  $K$  in  $S^3$ , let  $J_{K,n}(t)$  denote the  $n$ -colored Jones polynomial of  $K$  with the zero framing, normalized so that for the unknot  $U$ ,

$$J_{U,n}(t) = \frac{t^{2n} - t^{-2n}}{t^2 - t^{-2}}.$$

The colored Jones polynomial is a powerful quantum invariant that has many surprising connections to classical invariants. For example, the Melvin-Morton-Rozansky conjecture, proved in [1], states that the Alexander-Conway polynomial of a knot can be recovered from a certain limit of the colored Jones polynomials. The volume conjecture, raised in [7] and put in terms of the colored Jones polynomial in [11], is an important and still open conjecture relating the polynomial to the hyperbolic volume of the knot complement. In this paper, we investigate the  $AJ$ -conjecture, raised in [3], which relates recurrence relations of the sequence of colored Jones polynomials to the  $A$ -polynomial.

For every knot  $K$ , it was proven in [4] that the sequence of colored Jones polynomials  $J_{K,n}(t)$  satisfies a nontrivial recurrence relation. By defining  $J_{K,-n}(t) := -J_{K,n}(t)$  for  $n \in \mathbb{Z}$ , we can treat  $J_{K,n}(t)$  as a discrete function

$$J_{K,-}(t) : \mathbb{Z} \rightarrow \mathbb{Z}[t^{\pm 1}].$$

The quantum torus

$$\mathcal{T} = \mathbb{C}[t^{\pm 1}] \langle L^{\pm 1}, M^{\pm 1} \rangle / (LM - t^2 ML)$$

acts on the set of discrete functions  $f : \mathbb{Z} \rightarrow \mathbb{C}[t^{\pm 1}]$  by

$$(Mf)(n) := t^{2n} f(n), \quad (Lf)(n) := f(n+1).$$

A linear homogeneous recurrence relation  $\sum_{i=0}^d P_i(t, M) J_{K,n+i}(t) = 0$  then corresponds to a polynomial  $P(t, M, L) \in \mathcal{T}$  that is an annihilator of  $J_{K,n}(t)$  and vice versa. The set of all such annihilators

$$\mathcal{A}_K := \{P(t, M, L) \in \mathcal{T} \mid P(t, M, L) J_{K,n}(t) = 0\},$$

---

2010 MATHEMATICS SUBJECT CLASSIFICATION: PRIMARY 57M25  
KEYWORDS: COLORED JONES POLYNOMIAL, A-POLYNOMIAL, AJ-CONJECTURE

which is a left ideal of  $\mathcal{T}$ , is called the *recurrence ideal* of  $K$ . For every knot  $K$ , we know that  $\mathcal{A}_K$  is nontrivial.

The ring  $\mathcal{T}$  can be extended to a principal left ideal domain  $\tilde{\mathcal{T}}$  by including inverses of polynomials in  $t$  and  $M$ . In  $\tilde{\mathcal{T}}$ , we have a product defined by

$$f(t, M)L^a \cdot g(t, M)L^b = f(t, M)g(t, t^{2a}M)L^{a+b}$$

for any rational functions  $f(t, M), g(t, M)$  in  $\mathbb{C}(t, M)$ . The left ideal  $\tilde{\mathcal{A}}_K = \tilde{\mathcal{T}}\mathcal{A}_K$  is then generated by some nonzero polynomial in  $\tilde{\mathcal{T}}$ , and in particular, this generator can be chosen to be in  $\mathcal{A}_K$  and be of the form

$$\alpha_K(t, M, L) = \sum_{i=0}^d P_i L^i,$$

with  $d$  minimal and with  $P_1, \dots, P_d \in \mathbb{Z}[t, M]$  being relatively prime in  $\mathbb{Z}[t, M]$ . This polynomial  $\alpha_K$  is uniquely determined up to a sign and is called the (*normalized*) *recurrence polynomial* of  $K$ .

The  $A$ -polynomial was introduced in [2]. For a knot  $K$  in  $S^3$ , its  $A$ -polynomial  $A_K(M, L) \in \mathbb{Z}[M, L]$  is a two variable polynomial with no repeated factors and with relatively prime integer coefficients, which is uniquely associated to  $K$  up to a sign. Note that  $A_K(M, L)$  always contains the factor  $L - 1$ .

The  $AJ$ -conjecture states that for every knot  $K$ , its recurrence polynomial  $\alpha_K(t, M, L)$  evaluated at  $t = -1$  is equal to the  $A$ -polynomial of  $K$ , up to a factor of a polynomial in  $M$ . So far, it has been shown that torus knots, some classes of 2-bridge knots and pretzel knots, and most cabled knots over torus knots satisfy the conjecture; see [3], [14], [6], [8], [9], [15], [13].

In our previous work [13], we used explicit formulas to verify the  $AJ$ -conjecture for most cabled knots over torus knots. For the figure eight knot, its colored Jones polynomials and  $A$ -polynomial are more complicated, and the  $A$ -polynomials of the cabled knots over the figure eight knot are given in terms of the resultant of two polynomials, requiring a more theoretical connection.

**Theorem 1.1.** *The  $AJ$ -conjecture holds for each  $(r, s)$ -cabled knot  $C$  over the figure eight knot  $E$  when  $r > 4s$  or  $r < -4s$ .*

A cabling formula for  $A$ -polynomials of cabled knots in  $S^3$  is given in [12]. In particular, when  $C$  is the  $(r, s)$ -cabled knot over the figure eight knot  $E$  in  $S^3$ , its  $A$ -polynomial  $A_C(M, L)$  is given in terms of the  $A$ -polynomial  $A_E(M, L)$  of the figure eight knot. For a pair of relatively prime integers  $(r, s)$  with  $s \geq 2$ , define  $F_{(r,s)}(M, L) \in \mathbb{Z}[M, L]$  by:

$$F_{(r,s)}(M, L) := \begin{cases} M^{2r}L + 1, & \text{if } s = 2, r > 0, \\ L + M^{-2r}, & \text{if } s = 2, r < 0, \\ M^{2rs}L^2 - 1, & \text{if } s > 2, r > 0, \\ L^2 - M^{-2rs}, & \text{if } s > 2, r < 0 \end{cases}$$

Then

$$(1.1) \quad A_C(M, L) = (L - 1)F_{(r,s)}(M, L) \text{Red}\left(\text{Res}_\lambda\left(\frac{A_E(M^s, \lambda)}{\lambda - 1}, \lambda^s - L\right)\right),$$

where  $Red$  denotes the function reducing polynomials by eliminating repeated factors and  $Res_\lambda$  denotes the polynomial resultant eliminating the variable  $\lambda$ ; see Section 4.2 for a definition. The  $A$ -polynomial  $A_E(M, L)$  of the figure eight knot  $E$ , is

$$A_E(L, M) = (L - 1)(-L + LM^2 + M^4 + 2LM^4 + L^2M^4 + LM^6 - LM^8).$$

Meanwhile, its colored Jones polynomial, as given in [3] but in our normalized form, is

$$J_{E,n}(t) = \frac{t^{2n} - t^{-2n}}{t^2 - t^{-2}} \sum_{k=0}^{n-1} \prod_{i=1}^k ((t^{2n} - t^{-2n})^2 - (t^{2i} - t^{-2i})^2).$$

The figure eight knot  $E$  has an inhomogeneous recurrence polynomial  $(\tilde{\alpha}_E(t, M, L), b(t, M))$ , found in [3] and changed here to suit our normalization, given by

$$(1.2) \quad \begin{aligned} \tilde{\alpha}_E(t, M, L) J_{E,n}(t) &= (P_2(t, M)L^2 + P_1(t, M)L + P_0(t, M)) J_{E,n}(t) = b(t, M), \text{ where} \\ P_2(t, M) &= t^{10}M^4(-1 + t^4M^4) \\ P_1(t, M) &= -(-1 + t^4M^2)(1 + t^4M^2)(1 - t^4M^2 - t^4M^4 - t^{12}M^4 - t^{12}M^6 + t^{16}M^8) \\ P_0(t, M) &= t^6M^4(-1 + t^{12}M^4) \\ b(t, M) &= \frac{M(1+t^4M^2)(-1+t^4M^4)(-t^2+t^{14}M^4)}{t^2-t^{-2}}. \end{aligned}$$

Notice that any inhomogeneous recurrence relation gives rise to a homogeneous one, since if  $P(t, M, L)J_{K,n}(t) = b(t, M)$  for  $b(t, M) \neq 0$ , then

$$(L - 1)b(t, M)^{-1}P(t, M, L)J_{K,n}(t) = 0,$$

and by multiplying by a suitable polynomial in  $t$  and  $M$ , we can recover an annihilator in  $\mathcal{A}_K$ .

A cabling formula for the  $n$ -colored Jones polynomial of the  $(r, s)$ -cabled knot  $C$  over a knot  $K$  is given in [10] (see also [17]) which in our normalized form is:

$$(1.3) \quad J_{C,n}(t) = t^{-rs(n^2-1)} \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} t^{4rk(ks+1)} J_{K,2ks+1}(t).$$

We divide the proof Theorem 1.1 into two cases:

- (1)  $s = 2$ ;
- (2)  $s > 2$ .

In each case, we use a relation for the cabling formula (1.3) and an inhomogeneous recurrence relation of  $J_{E,n}(t)$  to obtain an annihilator of  $J_{C,n}(t)$ , and then proceed to prove that it is the recurrence polynomial  $\alpha_C(t, M, L)$  of  $C$  when  $r$  is not between  $-4s$  and  $4s$  by using formulas for the degrees of the colored Jones polynomials given in Section 2. For the case  $s = 2$ , we directly compute  $A_C(M, L)$  and  $\alpha_C(-1, M, L)$  to verify the  $AJ$ -conjecture, while for the case  $s > 2$ , we verify the  $AJ$ -conjecture by exploring the relationship between our annihilator and the resultant.

In practice, we find a minimal degree annihilator in  $\tilde{\mathcal{A}}_C$  of the form  $P = \sum_{i=0}^d P_i L^i$ , which is equal to the normalized recurrence polynomial up to a rational function  $C(t, M)$ , which is enough for the purpose of verifying the  $AJ$ -conjecture. Also note that changing the sign of  $r$  only changes the  $A$ -polynomial of  $C$  up to a power of  $M$ , which is of no consequence when checking up to a factor of a rational function in  $M$ .

**1.1. Acknowledgments.** The author would like to thank Xingru Zhang for numerous discussions and the referee for many helpful comments. Anh T. Tran [16] has independently obtained similar results for the  $(r, 2)$ -cables of the figure eight knot.

## 2. DEGREE FORMULAS AND PRELIMINARIES

For the rest of the paper, let  $E$  denote the figure eight knot and  $C$  the  $(r, s)$ -cabled knot over  $E$  unless specified otherwise. Also, we often write  $J_{K,n}$  rather than  $J_{K,n}(t)$  for brevity.

For a polynomial  $f(t) \in \mathbb{Z}[t^{\pm 1}]$ , let  $\ell[f]$  and  $\hbar[f]$  denote the lowest degree and highest degree of  $f$  in  $t$  respectively. For  $f(t), g(t) \in \mathbb{Z}[t^{\pm 1}]$ , these functions satisfy  $\ell[fg] = \ell[f] + \ell[g]$  and  $\hbar[fg] = \hbar[f] + \hbar[g]$ .

The degrees of the colored Jones polynomials of alternating knots are known [8, Proposition 2.1]. For a non-trivial knot  $K$  with reduced alternating diagram  $D$  having  $k$  crossings and writhe  $w$ , if  $n > 0$ , then

$$\begin{aligned}\hbar[J_{K,n}] &= k(n-1)^2 - w(n^2 - 1) + 2(n-1)s_+(D), \\ \ell[J_{K,n}] &= -k(n-1)^2 - w(n^2 - 1) - 2(n-1)s_-(D),\end{aligned}$$

where  $\hbar$  and  $\ell$  denote the highest and lowest degrees in  $t$ , respectively, and  $s_+(D)$ ,  $s_-(D)$  are the number of circles obtained by positively or negatively smoothing the crossings of  $D$ . Notice that  $s_-(D) + s_+(D) = k + 2$ , so  $2 \leq s_-(D), s_+(D) \leq k$ . In particular, the figure eight knot has a reduced alternating diagram with 4 crossings,  $s_+(D) = s_-(D) = 3$ , and zero writhe, so we have the following lemma.

**Lemma 2.1.** *Let  $E$  be the figure eight knot. Then for all  $n \neq 0$ ,*

$$\begin{aligned}\hbar[J_{E,n}] &= 4(|n| - 1)^2 + 6(|n| - 1) = 4n^2 - 2|n| - 2, \\ \ell[J_{E,n}] &= -4(|n| - 1)^2 - 6(|n| - 1) = -4n^2 + 2|n| + 2.\end{aligned}$$

The following lemma gives the degrees of  $J_{C,n}(t)$  for any cabled knot  $C$  over an alternating knot  $K$ .

**Lemma 2.2.** *Let  $C$  be the  $(r, s)$ -cable knot over a knot  $K$  with a reduced alternating diagram  $D$  with  $N$  crossings and  $s_-(D) = m$ ,  $s_+(D) = N + 2 - m$ . Then for  $n > N$ , we have*

$$\begin{aligned}\hbar[J_{C,n}] &= \begin{cases} (N-w)s^2n^2 + (2r - 2(-2 + m + r + w - N)s - 2(N-w)s^2)n & r > -(N-w)s \\ + (-2r + 2(-2 + m + r + w - N)s + (N-w)s^2), & \\ -rs(n^2 - 1) + \frac{1}{2}(1 - (-1)^{n-1})(s-2)(4 + r + (N-w)s - 2m), & r < -(N-w)s \end{cases} \\ \ell[J_{C,n}] &= \begin{cases} -(N+w)s^2n^2 + (2r - 2(r + m + w)s + 2(N+w)s^2)n & r < (N+w)s \\ + (-2r + 2(r + m + w)s - (N+w)s^2), & \\ -rs(n^2 - 1) + \frac{1}{2}(1 - (-1)^{n-1})(s-2)(r - (N+w)s + 2(N-m)), & r > (N+w)s \end{cases}\end{aligned}$$

In particular, for the cabled knot  $C$  over the figure eight knot, we set  $N = 4$ ,  $m = 3$ , and  $w = 0$ :

$$\begin{aligned} \hbar[J_{C,n}] &= \begin{cases} 4s^2n^2 + (2r + 6s - 2rs - 8s^2)n + (-2r - 6s + 2rs + 4s^2), & r > -4s \\ -rs(n^2 - 1) + \frac{1}{2}(1 - (-1)^{n-1})(s - 2)(-2 + r + 4s), & r < -4s \end{cases} \\ \ell[J_{C,n}] &= \begin{cases} -4s^2n^2 + (2r - 6s - 2rs + 8s^2)n + (-2r + 6s + 2rs - 4s^2), & r < 4s \\ -rs(n^2 - 1) + \frac{1}{2}(1 - (-1)^{n-1})(s - 2)(2 + r - 4s), & r > 4s \end{cases} \end{aligned}$$

The restriction  $n > N$  can usually be relaxed. For the figure eight knot case, the formulas hold for all  $n > 0$ .

*Proof.* We know from the cabling formula

$$\ell[J_{C,n}] = -rs(n^2 - 1) + \min\{\ell[J_{K,2sk+1}] + 4rk(sk + 1) \mid -\frac{n-1}{2} \leq k \leq \frac{n-1}{2}\}$$

where  $k$  is integer or half-integer valued. Let  $g(k) = \ell[J_{E,2sk+1}] + 4rk(sk + 1)$ . From the discussion above, we know

$$\begin{aligned} g(k) &= -N(|2sk + 1| - 1)^2 - w((2sk + 1)^2 - 1) - 2(|2sk + 1| - 1)m + 4rk(sk + 1) \\ &= \begin{cases} 4sk^2(r - (N + w)s) + 4k(r - (m + w)s), & k \geq 0 \\ 4sk^2(r - (N + w)s) + 4k(r - (2N - m + w)s) - 4(N - m), & k \leq -\frac{1}{2} \end{cases} \end{aligned}$$

Since  $r$  and  $s$  are relatively prime,  $r \neq (N + w)s$ , so each piece of  $g(k)$  is quadratic with critical points  $k = \frac{r - (m + w)s}{-2s(r - (N + w)s)}$  and  $k = \frac{r - (2N - m + w)s}{-2s(r - (N + w)s)}$  respectively. Notice that when  $r > (N + w)s$ , these critical points are local minima. Moreover,  $2 \leq m \leq N$ , so when  $r > (N + w)s$ , we have  $r > (m + w)s$ , so the first point is negative. Meanwhile, the second point is

$$\frac{r - (2N - m + w)s}{-2s(r - (N + w)s)} = \frac{r - (N + w)s}{-2s(r - (N + w)s)} + \frac{(-N + m)s}{-2s(r - (N + w)s)} = \frac{-1}{2s} + \frac{N - m}{2(r - (N + w)s)} > \frac{-1}{2},$$

so each component of  $g(k)$  is minimized at their endpoints  $k = 0$  and  $k = -1/2$ . Then  $g(0) = 0$  and  $g(-\frac{1}{2}) = (s - 2)(r - (N + w)s + 2(N - m))$ . The point  $k = 0$  is attained when  $n$  is odd, and  $k = -\frac{1}{2}$  occurs when  $n$  is even, so this gives us the formula

$$\ell[J_{C,n}] = -rs(n^2 - 1) + \frac{1}{2}(1 - (-1)^{n-1})(s - 2)(r - (N + w)s + 2(N - m))$$

when  $r > (N + w)s$ , for all  $n > 0$ .

In the case where  $r < (N + w)s$ ,  $g(k)$  must be minimized at either  $k = \frac{-(n-1)}{2}$  or  $k = \frac{n-1}{2}$ . For  $n > 1$ , we see that

$$g(\frac{n-1}{2}) - g(\frac{-(n-1)}{2}) = 4(r - (N + w)s)(n - 1) + 4(N - m),$$

which is negative for  $n > N$ , so  $g(\frac{n-1}{2})$  is smaller. Therefore

$$\ell[J_{C,n}] = -(N + w)s^2n^2 + (2r - 2(r + m + w)s + 2(N + w)s^2)n + (-2r + 2(r + m + w)s - (N + w)s^2).$$

The proof for  $\hbar[J_{C,n}]$  is similar.  $\diamond$

The following lemma is a generalization of Proposition 2.2 in [8].

**Lemma 2.3.** *Let  $f : \mathbb{Z} \rightarrow \mathbb{C}[t^{\pm 1}]$  be a discrete function. Suppose  $f$  satisfies the following conditions:*

- (1) *There exists an integer  $k$  and a nonzero  $c \in \mathbb{C}$  such that for all  $n$ ,  $f(-n) = cf(n+k)$ ,*
- (2)  *$f$  satisfies a nontrivial homogeneous recurrence relation, and*
- (3) *There exists an integer  $N$  such that for all  $n > N$ ,  $\hbar[f(n)] - \ell[f(n)]$  is not a linear function in  $n$ .*

*Then any nontrivial recurrence relation of  $f$  has order at least 2.*

*Proof.* Suppose  $f : \mathbb{Z} \rightarrow \mathbb{Z}[t^{\pm 1}]$  satisfies the given hypotheses, and assume toward a contradiction that there is homogeneous recurrence relation of  $f$  of order 1; that is, there exist nonzero  $P_1(t, t^{2n})$ ,  $P_0(t, t^{2n})$  such that  $P_1(t, t^{2n})f(n+1) + P_0(t, t^{2n})f(n) = 0$ . We can assume without loss of generality that  $P_1(t, t^{2n})$  and  $P_0(t, t^{2n})$  share no common factors. Then the annihilator ideal  $\tilde{\mathcal{A}}_f$  is nonzero and moreover is generated by  $P_1(t, M)L + P_0(t, M)$  since it is a principal left ideal.

Substituting  $-n$  for  $n$ , we know

$$\begin{aligned} 0 &= P_1(t, t^{-2n})f(-n+1) + P_0(t, t^{-2n})f(-n) \\ &= P_1(t, t^{-2n})cf(n+k-1) + P_0(t, t^{-2n})cf(n+k). \end{aligned}$$

Canceling  $c$  and shifting by replacing  $n$  with  $n-k+1$ , we have

$$0 = P_1(t, t^{-2n+2k-2})f(n) + P_0(t, t^{-2n+2k-2})f(n+1),$$

so  $P_0(t, t^{2k-2}M^{-1})L + P_1(t, t^{2k-2}M^{-1})$  is also in  $\tilde{\mathcal{A}}_f$ . Then this is a multiple of our generator, so since they both have  $L$ -degree 1, there is some  $B(t, M) \in \mathbb{C}(t, M)$  such that

$$B(t, M)(P_1(t, M)L + P_0(t, M)) = P_0(t, t^{2k-2}M^{-1})L + P_1(t, t^{2k-2}M^{-1}).$$

This implies  $B(t, M)P_1(t, M) = P_0(t, t^{2k-2}M^{-1})$  and  $B(t, M)P_0(t, M) = P_1(t, t^{2k-2}M^{-1})$ , so solving for  $B(t, M)$ ,

$$B(t, M) = \frac{P_0(t, t^{2k-2}M^{-1})}{P_1(t, M)} = \frac{P_1(t, t^{2k-2}M^{-1})}{P_0(t, M)}.$$

This tells us

$$P_0(t, M)P_0(t, t^{2k-2}M^{-1}) = P_1(t, M)P_1(t, t^{2k-2}M^{-1}).$$

Since  $P_0(t, M)$  and  $P_1(t, M)$  are relatively prime, we must have  $P_0(t, M)$  divides  $P_1(t, t^{2k-2}M^{-1})$ . Likewise,  $P_0(t, t^{2k-2}M^{-1})$  and  $P_1(t, t^{2k-2}M^{-1})$  are relatively prime, so we conclude that  $P_0(t, M) = P_1(t, t^{2k-2}M^{-1})$ . Substituting this into our original relation, we see

$$P_1(t, t^{2n})f(n+1) + P_1(t, t^{2k-2-2n})f(n) = 0$$

and thus

$$f(n+1) = -\frac{P_1(t, t^{2k-2-2n})}{P_1(t, t^{2n})}f(n).$$

Consider the degrees in  $t$  of both sides. We see that for  $n$  sufficiently large, the difference in breadths of  $P_1(t, t^{2n})$  and  $P_1(t, t^{2k-2-2n})$  is a constant  $K$ . This implies that  $\hbar[f(n)] - \ell[f(n)]$  is a linear function in  $n$  for large enough  $n$ , contrary to our assumption. We conclude that any recurrence relation of  $f$  must have order at least 2.  $\diamond$

3. CASE  $s = 2$ 

In this section, we prove the  $s = 2$  case of Theorem 1.1 in three steps: we first find a polynomial that annihilates  $J_{C,n}$ , then verify the AJ-conjecture by evaluating our polynomial at  $t = -1$ , and finally we prove that our annihilator is of minimal order and hence the recurrence polynomial.

**3.1. An annihilator of the colored Jones polynomial.** In [13, Lemma 3.1], the formula

$$J_{C,n+2} - t^{-4rsn-4rs} J_{C,n} = t^{2(r-rs)n-2rs+2r} J_{K,s(n+1)+1} - t^{2(-r-rs)n-2rs-2r} J_{K,s(n+1)-1}$$

is derived for any  $(r, s)$ -cabled knot  $C$  over any knot  $K$ . Rearranging and changing to operator notation, this is

$$(3.1) \quad (t^{2rs} M^{rs} L^2 - t^{-2rs} M^{-rs}) J_{C,n} = t^{2r} M^r J_{K,s(n+1)+1} - t^{-2r} M^{-r} J_{K,s(n+1)-1}.$$

When  $s = 2$ , it is computed in [13, Equation 6.1] that

$$(3.2) \quad J_{C,n+1} = -t^{-4rn-2r} J_{C,n} + t^{-2rn} J_{K,2n+1},$$

which can be rewritten as

$$(3.3) \quad (M^r L + t^{-2r} M^{-r}) J_{C,n} = J_{K,2n+1}.$$

Therefore, to find an annihilator of  $J_{C,n}$ , it is enough to find an annihilator of  $J_{K,2n+1}$ .

**Lemma 3.1.** *Let  $\tilde{\alpha}_E(t, M, L) = P_2(t, M)L^2 + P_1(t, M)L + P_0(t, M)$  be the inhomogeneous recurrence polynomial defined in equation (1.2). Then  $J_{E,2n+1}(t)$  has an inhomogeneous recurrence relation given by the polynomial  $Q(t, M, L) = Q_2(t, M)L^2 + Q_1(t, M)L + Q_0(t, M)$ , where*

$$\begin{aligned} Q_2(t, M) &= P_2(t, t^4 M^2) P_1(t, t^2 M^2) P_2(t, t^6 M^2), \\ Q_1(t, M) &= P_0(t, t^4 M^2) P_1(t, t^6 M^2) P_2(t, t^2 M^2) - P_1(t, t^6 M^2) P_1(t, t^2 M^2) P_1(t, t^4 M^2) \\ &\quad + P_2(t, t^4 M^2) P_1(t, t^2 M^2) P_0(t, t^6 M^2), \\ Q_0(t, M) &= P_0(t, t^4 M^2) P_1(t, t^6 M^2) P_0(t, t^2 M^2). \end{aligned}$$

*Proof.* Let  $\tilde{\alpha}_E(t, M, L) = P_2(t, M)L^2 + P_1(t, M)L + P_0(t, M)$  and  $b(t, M)$  be given as in equation (1.2). Changing  $M$  to  $t^{2n}$  for clarity, we have

$$P_2(t, t^{2n}) J_{E,n+2} + P_1(t, t^{2n}) J_{E,n+1} + P_0(t, t^{2n}) J_{E,n} = b(t, t^{2n}),$$

and substituting  $2n + 1$ ,  $2n + 2$ , and  $2n + 3$  for  $n$ , we have

$$(3.4) \quad \begin{aligned} P_2(t, t^{4n+2}) J_{E,2n+3} + P_1(t, t^{4n+2}) J_{E,2n+2} + P_0(t, t^{4n+2}) J_{E,2n+1} &= b(t, t^{4n+2}), \\ P_2(t, t^{4n+4}) J_{E,2n+4} + P_1(t, t^{4n+4}) J_{E,2n+3} + P_0(t, t^{4n+4}) J_{E,2n+2} &= b(t, t^{4n+4}), \\ P_2(t, t^{4n+6}) J_{E,2n+5} + P_1(t, t^{4n+6}) J_{E,2n+4} + P_0(t, t^{4n+6}) J_{E,2n+3} &= b(t, t^{4n+6}). \end{aligned}$$

A second degree inhomogeneous recurrence relation of  $J_{E,2n+1}(t)$  has the form

$$Q_2(t, t^{2n}) J_{E,2n+5} + Q_1(t, t^{2n}) J_{E,2n+3} + Q_0(t, t^{2n}) J_{E,2n+1} = B(t, t^{2n})$$

for some rational functions  $Q_i(t, t^{2n}), B(t, t^{2n})$ .

We claim that we can find a linear combination of the relations (3.4) that is of this form. That is, we want to solve

$$\sum_{j=0}^2 c_j \sum_{i=0}^2 P_i(t, t^{4n+2j+2}) J_{E, 2n+1+i+j} = \sum_{i=0}^2 Q_i(t, t^{2n}) J_{E, 2n+1+2i}$$

or equivalently

$$0 = \sum_{j=0}^2 c_j \sum_{i=0}^2 P_i(t, t^{4n+2j+2}) J_{E, 2n+1+i+j} - \sum_{i=0}^2 Q_i(t, t^{2n}) J_{E, 2n+1+2i}$$

for the unknown coefficients  $c_0, c_1, c_2 \in \mathbb{C}(t, M)$  and  $Q_0, Q_1, Q_2$ . It is enough to find the  $c_i$ 's and  $Q_i$ 's that make the coefficients on each  $J_{E, 2n+1+i+j}$  vanish. Then we have the following system of equations:

$$\begin{aligned} 0 &= (c_0 P_0(t, t^{4n+2}) - Q_0) J_{E, 2n+1} \\ 0 &= (c_0 P_1(t, t^{4n+2}) + c_1 P_0(t, t^{4n+4})) J_{E, 2n+2} \\ 0 &= (c_0 P_2(t, t^{4n+2}) + c_1 P_1(t, t^{4n+4}) + c_2 P_0(t, t^{4n+6}) - Q_1) J_{E, 2n+3} \\ 0 &= (c_1 P_2(t, t^{4n+4}) + c_2 P_1(t, t^{4n+6})) J_{E, 2n+4} \\ 0 &= (c_2 P_2(t, t^{4n+6}) - Q_2) J_{E, 2n+5} \end{aligned}$$

Setting the coefficients equal to zero, we form a  $5 \times 7$  augmented matrix:

$$\left[ \begin{array}{cccccc|c} P_0(t, t^{4n+2}) & 0 & 0 & -1 & 0 & 0 & 0 \\ P_1(t, t^{4n+2}) & P_0(t, t^{4n+4}) & 0 & 0 & 0 & 0 & 0 \\ P_2(t, t^{4n+2}) & P_1(t, t^{4n+4}) & P_0(t, t^{4n+6}) & 0 & -1 & 0 & 0 \\ 0 & P_2(t, t^{4n+4}) & P_1(t, t^{4n+6}) & 0 & 0 & 0 & 0 \\ 0 & 0 & P_2(t, t^{4n+6}) & 0 & 0 & -1 & 0 \end{array} \right]$$

where the columns correspond to  $c_0, c_1, c_2, Q_0, Q_1, Q_2$  respectively. We can simply row-reduce this:

$$\left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} -\frac{P_0(t, t^{4n+4})P_1(t, t^{4n+6})}{P_1(t, t^{4n+2})P_2(t, t^{4n+4})P_2(t, t^{4n+6})} \\ \frac{P_1(t, t^{4n+6})}{P_2(t, t^{4n+4})P_2(t, t^{4n+6})} \\ -\frac{1}{P_2(t, t^{4n+6})} \\ -\frac{P_0(t, t^{4n+2})P_0(t, t^{4n+4})P_1(t, t^{4n+6})}{P_1(t, t^{4n+2})P_2(t, t^{4n+4})P_2(t, t^{4n+6})} \\ \frac{P_1(t, t^{4n+2})P_1(t, t^{4n+4})P_1(t, t^{4n+6}) - P_0(t, t^{4n+4})P_1(t, t^{4n+6})P_2(t, t^{4n+2}) - P_0(t, t^{4n+6})P_1(t, t^{4n+2})P_2(t, t^{4n+4})}{P_1(t, t^{4n+2})P_2(t, t^{4n+4})P_2(t, t^{4n+6})} \end{array} \left[ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right]$$



Thus the system has a one-dimensional solution since  $P_1(t, t^{4n+2})$  is nonzero. Then we can choose  $Q_2 = P_1(t, t^{4n+2})P_2(t, t^{4n+4})P_2(t, t^{4n+6})$  to clear all of the denominators, which gives

$$\begin{aligned} Q_0 &= -P_0(t, t^{4n+2})P_0(t, t^{4n+4})P_1(t, t^{4n+6}), \\ Q_1 &= P_1(t, t^{4n+2})P_1(t, t^{4n+4})P_1(t, t^{4n+6}) - P_0(t, t^{4n+4})P_1(t, t^{4n+6})P_2(t, t^{4n+2}) \\ &\quad - P_0(t, t^{4n+6})P_1(t, t^{4n+2})P_2(t, t^{4n+4}), \\ c_0 &= P_0(t, t^{4n+4})P_1(t, t^{4n+6}), \\ c_1 &= -P_1(t, t^{4n+2})P_1(t, t^{4n+6}), \\ c_2 &= P_1(t, t^{4n+2})P_2(t, t^{4n+4}). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \sum_{i=0}^2 Q_i(t, t^{2n}) J_{E, 2n+1+2i} &= \sum_{j=0}^2 c_j \sum_{i=0}^2 P_i(t, t^{4n+2j+2}) J_{E, 2n+1+i+j} \\ &= \sum_{j=0}^2 c_j b(t, t^{4n+2j+2}). \end{aligned}$$

We can see that  $B(t, M) := \sum_{j=0}^2 c_j b(t, t^{2j+2}M^2)$  is nonzero by evaluating the limit as  $t$  approaches  $-1$  of  $(t^2 - t^{-2})B(t, M)$ :

$$\begin{aligned} \lim_{t \rightarrow -1} (t^2 - t^{-2})B(t, M) &= \lim_{t \rightarrow -1} (t^2 - t^{-2})(P_0(t, t^4M^2)P_1(t, t^6M^2)b(t, t^2M^2) \\ &\quad - P_1(t, t^2M^2)P_1(t, t^6M^2)b(t, t^4M^2) + P_2(t, t^2M^2)P_1(t, t^4M^2)b(t, t^6M^2)) \\ &= (M(-1 + M^4)(-1 + M^4))(P_0(1, M^2)P_1(1, M^2) - P_1(1, M^2)^2 \\ &\quad + P_1(1, M^2)P_2(1, M^2)) \\ &= (-1 + M)^6 M^2 (1 + M)^6 (1 + M^2)^6 (1 - M + M^2)(1 + M + M^2)(1 + M^4)^5 \\ &\quad (-1 + M^2 - M^4)(1 - M^4 - 2M^8 - M^{12} + M^{16}), \end{aligned}$$

which is indeed nonzero. Therefore  $Q(t, M, L)$  is an inhomogeneous recurrence polynomial of  $J_{E, 2n+1}$ .  $\diamond$

Notice that our inhomogeneous recurrence relation  $Q(t, M, L)(M^r L + t^{-2r} M^{-r})J_{C, n} = B(t, M)$  found in Lemma 3.1 can be made into the homogeneous one

$$R(t, M, L)J_{C, n} = (L - 1)B(t, M)^{-1}Q(t, M, L)(M^r L + t^{-2r} M^{-r})J_{C, n} = 0.$$

Assuming  $R(t, M, L)$  is the recurrence polynomial of  $C$ , we can check the  $AJ$ -conjecture by evaluating at  $t = -1$ . We can directly compute the  $A$ -polynomial of  $C$  from equation (1.1):

$$\begin{aligned} A_C(M, L) &= (L - 1)(M^{2r}L + 1)(-L + 2LM^4 + 3LM^8 - 2LM^{12} + M^{16} - 6LM^{16} + L^2M^{16} - 2LM^{20} \\ &\quad + 3LM^{24} + 2LM^{28} - LM^{32}). \end{aligned}$$

And so

(3.5)

$$\begin{aligned} R(-1, M, L) = & (-1 + M)^3(1 + M)^3(1 + M^2)^3(1 + M^4)^3(1 - M^4 - 2M^8 - M^{12} + M^{16}) \\ & \times B(-1, M)^{-1}(L - 1)(M^r L + M^{-r}) \\ & \times (L - 2LM^4 - 3LM^8 + 2LM^{12} - M^{16} + 6LM^{16} - L^2M^{16} + 2LM^{20} - 3LM^{24} \\ & - 2LM^{28} + LM^{32}) \end{aligned}$$

which, up to a factor of an element in  $\mathbb{C}(M)$ , is equal to the  $A$ -polynomial of  $C$ .

**3.2. Minimal degree of the recurrence relation.** In this section, we prove that the operator  $R(t, M, L)$  given in the previous section has minimal  $L$ -degree by showing that no annihilator of  $J_{C,n}$  has  $L$ -degree less than 4.

By Lemma 2.3, a recurrence relation for  $J_{C,n}$  has  $L$ -degree at least 2.

Suppose the recurrence polynomial of  $C$  has  $L$ -degree 2 or 3. Then there are relatively prime Laurent polynomials  $D_0, \dots, D_3$  in  $t$  and  $M$  ( $D_3$  possibly 0) such that

$$D_3 J_{C,n+3} + D_2 J_{C,n+2} + D_1 J_{C,n+1} + D_0 J_{C,n} = 0.$$

Using equation (3.1) to reduce  $J_{C,n+2}$  and  $J_{C,n+3}$ ,

$$\begin{aligned} 0 = & D_3 J_{C,n+3} + D_2 J_{C,n+2} + D_1 J_{C,n+1} + D_0 J_{C,n} \\ = & D_3(-t^{-4r(n+2)-2r} J_{C,n+2} + t^{-2r(n+2)} J_{E,2n+5}) + D_2 J_{C,n+2} + D_1 J_{C,n+1} + D_0 J_{C,n} \\ = & D_3 t^{-2rn-4r} J_{E,2n+5} + (D_2 - D_3 t^{-4rn-10r})(-t^{-4r(n+1)-2r} J_{C,n+1} + t^{-2r(n+1)} J_{E,2n+3}) \\ & + D_1 J_{C,n+1} + D_0 J_{C,n} \\ = & D_3 t^{-2rn-4r} J_{E,2n+5} + (D_2 t^{-2rn-2r} - D_3 t^{-6rn-12r}) J_{E,2n+3} \\ & + (D_1 - D_2 t^{-4rn-6r} + D_3 t^{-8rn-16r})(-t^{-4rn-2r} J_{C,n} + t^{-2rn} J_{E,2n+1}) + D_0 J_{C,n} \\ = & D_3 t^{-2rn-4r} J_{E,2n+5} + (D_2 t^{-2rn-2r} - D_3 t^{-6rn-12r}) J_{E,2n+3} \\ & + (D_1 t^{-2rn} - D_2 t^{-6rn-6r} + D_3 t^{-10rn-16r}) J_{E,2n+1} \\ & + (D_0 - D_1 t^{-4rn-2r} + D_2 t^{-8rn-8r} - D_3 t^{-12rn-18r}) J_{C,n}, \end{aligned}$$

and since  $Q_2 J_{E,2n+5} + Q_1 J_{E,2n+3} + Q_0 J_{E,2n+1} = B$ , we have  $J_{E,2n+5} = \frac{B}{Q_2} - \frac{Q_1}{Q_2} J_{E,2n+3} - \frac{Q_0}{Q_2} J_{E,2n+1}$ , and so

$$\begin{aligned} 0 = & D_3 t^{-2rn-4r} \left( \frac{B}{Q_2} - \frac{Q_1}{Q_2} J_{E,2n+3} - \frac{Q_0}{Q_2} J_{E,2n+1} \right) + (D_2 t^{-2rn-2r} - D_3 t^{-6rn-12r}) J_{E,2n+3} \\ & + (D_1 t^{-2rn} - D_2 t^{-6rn-6r} + D_3 t^{-10rn-16r}) J_{E,2n+1} \\ & + (D_0 - D_1 t^{-4rn-2r} + D_2 t^{-8rn-8r} - D_3 t^{-12rn-18r}) J_{C,n} \\ = & D_3 \frac{B}{Q_2} t^{-2rn-4r} + (D_2 t^{-2rn-2r} - D_3 t^{-6rn-12r} - D_3 \frac{Q_1}{Q_2} t^{-2rn-4r}) J_{E,2n+3} \\ & + (D_1 t^{-2rn} - D_2 t^{-6rn-6r} + D_3 t^{-10rn-16r} - D_3 \frac{Q_0}{Q_2} t^{-2rn-4r}) J_{E,2n+1} \\ & + (D_0 - D_1 t^{-4rn-2r} + D_2 t^{-8rn-8r} - D_3 t^{-12rn-18r}) J_{C,n}, \end{aligned}$$

and multiplying everything by  $Q_2$ , we have

$$0 = D'_0 + D'_1 J_{E,2n+3} + D'_2 J_{E,2n+1} + D'_3 J_{C,n}$$

where the  $D'_i$  are Laurent polynomials in  $t$  and  $M$  given by

$$\begin{aligned} D'_0 &= BD_3 t^{-2rn-4r}, \\ D'_1 &= Q_2 D_2 t^{-2rn-2r} - Q_2 D_3 t^{-6rn-12r} - Q_1 D_3 t^{-2rn-4r}, \\ D'_2 &= Q_2 D_1 t^{-2rn} - Q_2 D_2 t^{-6rn-6r} + Q_2 D_3 t^{-10rn-16r} - Q_0 D_3 t^{-2rn-4r}, \\ D'_3 &= Q_2 D_0 - Q_2 D_1 t^{-4rn-2r} + Q_2 D_2 t^{-8rn-8r} - Q_2 D_3 t^{-12rn-18r}. \end{aligned}$$

Notice that if  $D'_0 = D'_1 = D'_2 = D'_3 = 0$ , then it follows that  $D_0 = D_1 = D_2 = D_3 = 0$  as well. If the recurrence polynomial has L-degree 2, then  $D_3 = 0$  and thus  $D'_0 = 0$ . The following lemma rules out this possibility.

**Lemma 3.2.** *When  $r > 8$  or  $r < -8$ , if  $D'_0 = 0$ , then  $D'_i = 0$  for  $i = 1, 2, 3$  as well.*

*Proof.* Suppose  $D'_3 \neq 0$  and  $r > 8$ . Then the lowest degree in  $t$  of  $D'_3 J_{C,n}$  is  $\ell[D'_3] + \ell[J_{C,n}]$ . This term must vanish in the sum, so it must be canceled by another nonzero term, hence there must be another  $D'_i \neq 0$ . Then we must have

$$\ell[D'_3] + \ell[J_{C,n}] = \ell[D'_1 J_{E,2n+3} + D'_2 J_{E,2n+1}] \geq \min(\ell[D'_1] + \ell[J_{E,2n+3}], \ell[D'_2] + \ell[J_{E,2n+1}])$$

due to possible cancellation if  $D'_1 J_{E,2n+3}$  and  $D'_2 J_{E,2n+1}$  have the same lowest degree, and we consider  $\ell[0] = \infty$ .

For sufficiently large  $n$ ,  $\ell[D'_i]$  is a linear function in  $n$ . By Lemma 2.1, we have  $\ell[J_{E,2n+1}] = -16n^2 - 12n$  and  $\ell[J_{E,2n+3}] = -16n^2 - 48n - 28$ , and by Lemma 2.2, we know  $\ell[J_{C,n}] = -2rn^2 + 2r$ .

Suppose  $\min(\ell[D'_1] + \ell[J_{E,2n+3}], \ell[D'_2] + \ell[J_{E,2n+1}]) = \ell[D'_2] + \ell[J_{E,2n+1}]$ . Then for large enough  $n$ ,

$$\begin{aligned} \ell[D'_3] + \ell[J_{C,n}] &\geq \ell[D'_2] + \ell[J_{E,2n+1}], \text{ so} \\ \ell[D'_3] - \ell[D'_2] &\geq \ell[J_{E,2n+1}] - \ell[J_{C,n}] \\ &= -16n^2 - 12n - (-2rn^2 + 2r) \\ &= (-16 + 2r)n^2 - 12n - 2r, \end{aligned}$$

and since  $r > 8$ ,  $-16 + 2r > 0$ , so since the right hand side is quadratic in  $n$ , it will eventually be larger than the left hand side, which is only linear in  $n$ . This is a contradiction.

Likewise, if  $\min(\ell[D'_1] + \ell[J_{E,2n+3}], \ell[D'_2] + \ell[J_{E,2n+1}]) = \ell[D'_1] + \ell[J_{E,2n+3}]$ , we reach the same contradiction and conclude that  $D'_3 = 0$ . Similarly, if  $r < -8$ , we look at  $\ell[D'_3 J_{C,n}]$  and also conclude  $D'_3 = 0$ .

Now that  $D'_3 = 0$ , we have  $0 = D'_1 J_{E,2n+3} + D'_2 J_{E,2n+1}$ . Suppose  $D'_1 \neq 0$ . Then  $D'_2 \neq 0$  as well, since  $J_{E,2n+3}$  is not the zero function. Then we have a first order homogeneous recurrence relation of  $J_{E,2n+1}$ . If we can show that  $J_{E,2n+1}$  satisfies the hypotheses of Lemma 2.3, we will arrive at a contradiction. We have some recurrence relation for  $J_{E,2n+1}$  and it is easy to see

that  $\hbar[J_{E,2n+1}] - \ell[J_{E,2n+1}]$  is quadratic in  $n$ . Recalling that  $J_{E,-n} = -J_{E,n}$ , we have

$$J_{E,-2n+1} = -J_{E,2n-1} = -J_{E,2n+1-2},$$

so by Lemma 2.3,  $J_{E,2n+1}$  cannot have a first order homogeneous recurrence relation. Therefore,  $D'_1 = D'_2 = D'_3 = 0$ , as needed.  $\diamond$

We now know that the recurrence polynomial of  $C$  does not have degree 2.

**Lemma 3.3.** *When  $r > 8$  or  $r < -8$ , we have  $D'_i = 0$  for  $i = 0, 1, 2, 3$ .*

*Proof.* The proof that  $D'_3 = 0$  is the same as in the proof of Lemma 3.2, noting that the polynomial  $D'_0$  has lowest degree in  $t$  which is only linear in  $n$ , so  $\ell[D'_3 J_{C,n}] \neq \ell[D'_0]$  since  $\ell[J_{C,n}]$  is quadratic in  $n$ .

Since  $D'_3 = 0$ , we have  $0 = D'_0 + D'_1 J_{E,2n+3} + D'_2 J_{E,2n+1}$ . If  $D'_0 = 0$ , then we are done by Lemma 3.2, so assume for the sake of contradiction that  $D'_0 \neq 0$ . Then we have an inhomogeneous recurrence relation of  $J_{E,2n+1}$  of L-degree 1. This gives rise to a homogeneous recurrence of L-degree 2

$$\begin{aligned} 0 &= (L - 1)D'_0{}^{-1}(D'_1 L + D'_2)J_{E,2n+1} \\ &= (LD'_0{}^{-1}D'_1 L + LD'_0{}^{-1}D'_2 - D'_0{}^{-1}D'_1 L - D'_0{}^{-1}D'_2)J_{E,2n+1} \\ &= (D'_0{}^{-1}(t, t^2 M)D'_1(t, t^2 M)L^2 + (D'_0{}^{-1}(t, t^2 M)D'_2(t, t^2 M) - D'_0{}^{-1}(t, M)D'_1(t, M))L \\ &\quad - D'_0{}^{-1}(t, M)D'_2(t, M))J_{E,2n+1}, \end{aligned}$$

and multiplying on the left by  $D'_0(t, M)D'_0(t, t^2 M)$  gives

$$0 = (D'_0(t, M)D'_1(t, t^2 M)L^2 + (D'_0(t, M)D'_2(t, t^2 M) - D'_0(t, t^2 M)D'_1(t, M))L - D'_0(t, t^2 M)D'_2(t, M))J_{E,2n+1}.$$

Recalling that  $J_{E,2n+1} = (M^r L + t^{-2r} M^{-r})J_{C,n}$ , we get a homogeneous recurrence of  $J_{C,n}$  with the annihilator

$$\begin{aligned} S(t, M, L) &:= (D'_0(t, M)D'_1(t, t^2 M)L^2 + (D'_0(t, M)D'_2(t, t^2 M) - D'_0(t, t^2 M)D'_1(t, M))L \\ &\quad - D'_0(t, t^2 M)D'_2(t, M))(M^r L + t^{-2r} M^{-r}). \end{aligned}$$

We want to evaluate at  $t = -1$ , but it is possible that some  $D'_i(-1, M) = 0$ . Then for  $i = 0, 1, 2$ , we have  $D'_i(t, M) = (1+t)^{k_i} D''_i(t, M)$  for some Laurent polynomial  $D''_i(t, M)$  and minimal  $k_i \geq 0$  such that  $D''_i(-1, M) \neq 0$ . Hence

$$\begin{aligned} 0 &= ((1+t)^{k_0+k_1} D''_0(t, M)D''_1(t, t^2 M)L^2 + (1+t)^{k_0+k_2} D''_0(t, M)D''_2(t, t^2 M)L \\ &\quad - (1+t)^{k_0+k_1} D''_0(t, t^2 M)D''_1(t, M)L - (1+t)^{k_0+k_2} D''_0(t, t^2 M)D''_2(t, M))(M^r L + t^{-2r} M^{-r})J_{C,n} \\ &= (1+t)^{k_0} ((1+t)^{k_1} D''_0(t, M)D''_1(t, t^2 M)L^2 + (1+t)^{k_2} D''_0(t, M)D''_2(t, t^2 M)L \\ &\quad - (1+t)^{k_1} D''_0(t, t^2 M)D''_1(t, M)L - (1+t)^{k_2} D''_0(t, t^2 M)D''_2(t, M))(M^r L + t^{-2r} M^{-r})J_{C,n}, \end{aligned}$$

and we can cancel  $(1+t)^{k_0}$  and likewise any other common factors if  $k_1 > 0$  and  $k_2 > 0$ . Therefore we can assume without loss of generality that  $k_0 = 0$  and at least one of  $k_1$  or  $k_2 = 0$ . We check the cases.

**Case 1.**  $k_1 = 0$  and  $k_2 = 0$ :

Evaluating the annihilator  $S$  at  $t = -1$ , we have

$$\begin{aligned} S(-1, M, L) &= (D_0''(-1, M)D_1''(-1, M)L^2 + D_0''(-1, M)D_2''(-1, M)L \\ &\quad - D_0''(-1, M)D_1''(-1, M)L - D_0''(-1, M)D_2''(-1, M))(M^rL + M^{-r}) \\ &= D_0''(-1, M)(L - 1)(M^rL + M^{-r})(D_1''(-1, M)L + D_2''(-1, M)). \end{aligned}$$

Since the recurrence polynomial of  $C$  has  $L$ -degree 3 by assumption, we know that  $S(t, M, L)$  is the recurrence polynomial up to a factor of a Laurent polynomial in  $t$  and  $M$ , so over the field  $\mathbb{C}(M)$ ,  $S(-1, M, L)$  must divide our degree 4 homogenous annihilator  $R(t, M, L)$  of  $J_{C,n}$  valued at  $t = -1$  found in equation 3.5. This means we must have  $D_1''(-1, M)L + D_2''(-1, M)$  divides

$$L - 2LM^4 - 3LM^8 + 2LM^{12} - M^{16} + 6LM^{16} - L^2M^{16} + 2LM^{20} - 3LM^{24} - 2LM^{28} + LM^{32}.$$

But this is irreducible over  $\mathbb{C}(M)$ , so  $D_1''(-1, M)L + D_2''(-1, M) = 0$ , thus  $D_1''(-1, M) = 0$ , which is a contradiction.

**Case 2.**  $k_1 = 0$  and  $k_2 > 0$ :

Here, we have

$$\begin{aligned} S(-1, M, L) &= (D_0''(-1, M)D_1''(-1, M)L^2 - D_0''(-1, M)D_1''(-1, M)L)(M^rL + M^{-r}) \\ &= D_0''(-1, M)D_1''(-1, M)(L)(L - 1)(M^rL + M^{-r}), \end{aligned}$$

and so  $L$  must divide the irreducible factor of our degree 4 annihilator of  $J_{C,n}$ , which is a contradiction.

**Case 3.**  $k_1 > 0$  and  $k_2 = 0$ :

This time,

$$S(-1, M, L) = D_0''(-1, M)D_2''(-1, M)(L - 1)(M^rL + M^{-r})$$

which has  $L$ -degree 2. Since  $S(t, M, L)$  is, up to a factor of a Laurent polynomial in  $t$  and  $M$ , the recurrence polynomial of  $C$ , there must be some  $P(t, M, L)$  in  $\tilde{\mathcal{T}}$  such that  $R(t, M, L) = P(t, M, L)S(t, M, L)$ , so  $P$  has  $L$ -degree 1. However,  $R(-1, M, L) = P(-1, M, L)S(-1, M, L)$ , where  $R$  has  $L$ -degree 4 while the right-hand side of the equation has  $L$ -degree 3. This is a contradiction.

In each case, we arrive at a contradiction, and conclude that  $D'_0 = D'_1 = D'_2 = D'_3 = 0$ .  $\diamond$

Therefore, our annihilator  $R(t, M, L)$  is of minimal degree and hence the recurrence polynomial of  $J_{C,n}$  up to normalization when  $r > 8$  or  $r < -8$ .

#### 4. CASE: $s > 2$

We prove the  $s > 2$  case of Theorem 1.1 in the same three steps as before.

4.1. **The Annihilator.** Recall equation (3.1):

$$(t^{2rs}M^{rs}L^2 - t^{-2rs}M^{-rs})J_{C,n} = t^{2r}M^r J_{E,s(n+1)+1} - t^{-2r}M^{-r} J_{E,s(n+1)-1}.$$

Define the sequence  $T_n$  to be the right hand side of this equation:

$$(4.1) \quad T_n = t^{2r}M^r J_{E,s(n+1)+1} - t^{-2r}M^{-r} J_{E,s(n+1)-1}.$$

Then to find a recurrence relation of  $J_{C,n}$ , it is enough to find an inhomogenous recurrence relation for  $T_n$ .

**Proposition 4.1.** *There exists a polynomial  $Q(t, M, L)$  of  $L$ -degree 2 which satisfies  $Q(t, M, L)T_n = B(t, M)$  for some  $B(t, M) \in \mathbb{C}(t, M)$ .*

*Proof.* Any second order inhomogeneous recurrence relation for  $T_n$  looks like

$$\begin{aligned} B(t, M) &= \sum_{i=0}^2 Q_i(t, M)T_{n+i} \\ &= Q_0(t, M)(t^{2r}M^r J_{E,s(n+1)+1} - t^{-2r}M^{-r} J_{E,s(n+1)-1}) \\ &\quad + Q_1(t, M)(t^{4r}M^r J_{E,s(n+2)+1} - t^{-4r}M^{-r} J_{E,s(n+2)-1}) \\ &\quad + Q_2(t, M)(t^{6r}M^r J_{E,s(n+3)+1} - t^{-6r}M^{-r} J_{E,s(n+3)-1}). \end{aligned}$$

Recall that we have an inhomogeneous recurrence relation for  $J_{E,n+i}$  given in equation (1.2):

$$\sum_{i=0}^2 P_i(t, t^{2n})J_{E,n+i} = b(t, t^{2n}).$$

By substituting  $s(n+1) - 1 + k$ , with  $0 \leq k \leq 2s$ , for  $n$  in equation (1.2), we shift the relation to

$$\sum_{i=0}^2 P_i(t, t^{2(s(n+1)-1+k)})J_{E,s(n+1)-1+k+i} = b(t, t^{2(s(n+1)-1+k)}),$$

or changing  $t^{2n}$  to  $M$ ,

$$\sum_{i=0}^2 P_i(t, t^{2(s-1+k)}M^s)J_{E,s(n+1)-1+k+i} = b(t, t^{2(s-1+k)}M^s).$$

As in the  $s = 2$  case, we take a linear combination over  $\mathbb{C}(t, M)$  of these relations and aim to solve

$$(4.2) \quad \sum_{k=0}^{2s} c_k \sum_{i=0}^2 P_i(t, t^{2(s-1+k)}M^s)J_{E,s(n+1)-1+k+i} = \sum_{i=0}^2 Q_i(t, M)T_{n+i}$$

for the unknown coefficients  $c_0, \dots, c_{2s}$  and  $Q_0, Q_1, Q_2$ . With these in hand, we will have

$$\begin{aligned} \sum_{i=0}^2 Q_i(t, M)T_{n+i} &= \sum_{k=0}^{2s} c_k \sum_{i=0}^2 P_i(t, t^{2(s-1+k)}M^s)J_{E,s(n+1)-1+k+i} \\ &= \sum_{k=0}^{2s} c_k b(t, t^{2(s-1+k)}M^s) \in \mathbb{C}(t, M), \end{aligned}$$

so  $\sum_{i=0}^2 Q_i(t, M)L^i$  will be a polynomial giving a recurrence relation for  $T_n$ .

To solve equation (4.2), we replace  $T_{n+i}$  with its definition and see that this equation is equivalent to

$$\begin{aligned} 0 &= \sum_{k=0}^{2s} c_k \left( \sum_{i=0}^2 P_i(t, t^{2(s-1+k)} M^s) J_{E, s(n+1)-1+k+i} \right) - \sum_{i=0}^2 Q_i(t, M) T_{n+i} \\ &= \sum_{k=0}^{2s} c_k \left( \sum_{i=0}^2 P_i(t, t^{2(s-1+k)} M^s) J_{E, s(n+1)-1+k+i} \right) \\ &\quad - \sum_{i=0}^2 Q_i(t, M) (t^{(1+i)2r} M^r J_{E, s(n+1+i)+1} - t^{-(1+i)2r} M^{-r} J_{E, s(n+1+i)-1}). \end{aligned}$$

By setting the coefficients of each  $J_{E, s(n+1)-1+k}$  equal to zero, we obtain a linear system of equations over the field  $\mathbb{C}(t, M)$ , with  $2s + 3$  equations (the indices range from  $s(n+1) - 1$  to  $s(n+3) + 1$ ) and  $2s + 4$  unknowns. We obtain a  $(2s + 3) \times (2s + 4)$  matrix of coefficients, where we shorten the notation  $P_i(t, M)$  to  $P_i(M)$ :

(4.3)

$$A = \left[ \begin{array}{cccccc|ccc} P_0(t^{2s-2}M^s) & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & t^{-2r}M^{-r} \\ P_1(t^{2s-2}M^s) & P_0(t^{2s}M^s) & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ P_2(t^{2s-2}M^s) & P_1(t^{2s}M^s) & P_0(t^{2s+2}M^s) & \cdots & 0 & 0 & 0 & 0 & 0 & -t^{2r}M^r \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & P_2(t^{6s-6}M^s) & P_1(t^{6s-4}M^s) & P_0(t^{6s-2}M^s) & t^{-6r}M^{-r} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & P_2(t^{6s-4}M^s) & P_1(t^{6s-2}M^s) & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & P_2(t^{6s-2}M^s) & -t^{6r}M^r & 0 & 0 \end{array} \right]$$

where the columns are arranged corresponding to the order  $(c_0, c_1, \dots, c_{2s}, Q_2, Q_1, Q_0)$ , the rows are arranged in order of increasing index of  $J_{E, s(n+1)-1+k}$ , and the second-to-last column, corresponding to  $Q_1$ , contains  $t^{-4r}M^{-r}$  and  $-t^{4r}M^r$  in the  $s+1$  and  $s+3$  positions respectively and is zero everywhere else. We claim that this matrix has rank  $2s + 3$ , which will guarantee that our polynomial has degree 2 rather than degree 1.

We wish to row-reduce  $A$  enough to get a nonzero entry in the  $(2s + 3, 2s + 3)$  position. It is enough to do this setting  $t = -1$ . We can accomplish this in  $2s + 1$  steps, using the row operations

$$\mathcal{O}_i = \left\{ R_{i+1} \mapsto R_{i+1} - \frac{P_1(M^s)}{P_0(M^s)} R_i, R_{i+2} \mapsto R_{i+2} - \frac{P_2(M^s)}{P_0(M^s)} R_i \right\}$$

for  $1 \leq i \leq 2s + 1$ . Performing these in succession leaves us with the following matrix (where we abbreviate  $P_i(-1, M^s)$  to  $P_i(M^s)$ ):

$$B = \left[ \begin{array}{cccccc|ccc} P_0(M^s) & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & M^{-r} \\ 0 & P_0(M^s) & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \frac{-P_1(M^s)}{P_0(M^s)} M^r \\ 0 & 0 & P_0(M^s) & \cdots & 0 & 0 & 0 & 0 & 0 & -M^r - \frac{P_2(M^s)}{P_0(M^s)} M^{-r} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & P_0(M^s) & M^{-r} & x_{2s+1} & y_{2s+1} \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \frac{-P_1(M^s)}{P_0(M^s)} M^{-r} & x_{2s+2} & y_{2s+2} \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -M^r - \frac{P_2(M^s)}{P_0(M^s)} M^{-r} & x_{2s+3} & y_{2s+3} \end{array} \right]$$

For some  $x_i$ 's and  $y_i$ 's, which we will discuss in a moment. It is easy to verify that  $P_1(-1, M^s) \neq 0$ , so we can further reduce this to

$$\left[ \begin{array}{cccccc|ccc} P_0(M^s) & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & M^{-r} \\ 0 & P_0(M^s) & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \frac{-P_1(M^s)}{P_0(M^s)}M^{-r} \\ 0 & 0 & P_0(M^s) & \cdots & 0 & 0 & 0 & 0 & 0 & -M^r - \frac{P_2(M^s)}{P_0(M^s)}M^{-r} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & P_0(M^s) & 0 & x_{2s+1} + \frac{P_0(M^s)}{P_1(M^s)}x_{2s+2} & y'_{2s+1} \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \frac{-P_1(M^s)}{P_0(M^s)}M^{-r} & x_{2s+2} & y'_{2s+2} \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & x'_{2s+3} & y'_{2s+3} \end{array} \right]$$

where  $x'_{2s+3} = x_{2s+3} - \frac{P_0(M^s)}{P_1(M^s)}(M^{2r} + \frac{P_2(M^s)}{P_0(M^s)}x_{2s+2})$  (and similarly define  $y'_{2s+3}$ ). We now claim  $x'_{2s+3} \neq 0$ , which implies that this matrix has rank  $2s+3$ .

In the matrix  $B$  above, we denoted the entries of the  $2s+2$  column as  $x_1, \dots, x_{2s+3}$ . We have  $x_i = 0$  for  $1 \leq i \leq s$  and  $x_{s+1} = M^{-r}$ . Notice that due to the row operations, we have

$$\begin{aligned} x_{s+2} &= \frac{-P_1(M^s)}{P_0(M^s)}M^{-r}, \\ x_{s+3} &= -M^r + \frac{P_1(M^s)^2}{P_0(M^s)^2}M^{-r} - \frac{P_2(M^s)}{P_0(M^s)}M^{-r}, \\ x_i &= \frac{P_2(M^s)}{P_0(M^s)}x_{i-2} - \frac{P_1(M^s)}{P_0(M^s)}x_{i-1}, \quad s+4 \leq i \leq 2s+3 \end{aligned}$$

For a nonzero Laurent polynomial  $f \in \mathbb{C}[t, M]^*$ , define  $\mu(f)$  to be the maximum degree in  $M$  of  $f$ , and extend to rational functions  $f/g \in \mathbb{C}(t, M)^*$  by defining  $\mu(f/g) = \mu(f) - \mu(g)$ . This is well-defined and satisfies:

- $\mu(f \cdot g) = \mu(f) + \mu(g)$ .
- $\mu(f + g) = \max(\mu(f), \mu(g))$  if  $\mu(f) \neq \mu(g)$ .

In particular, we have  $\mu(P_0(1, M^s)) = \mu(P_2(1, M^s)) = 8s$  and  $\mu(P_1(1, M^s)) = 12s$ . We will show that  $x'_{2s+3} \neq 0$  by showing  $\mu(x'_{2s+3})$  is defined.

We have  $\mu(x_{s+1}) = -r$ ,  $\mu(x_{s+2}) = \mu(P_1(M^s)) - \mu(P_2(M^s)) + \mu(M^{-r}) = 4s - r$ ,  $\mu(x_{s+3}) = \max(r, 8s - r)$ , and  $\mu(x_i) = \max(\mu(x_{i-2}), 4s + \mu(x_{i-1}))$ . Let us examine the cases.

**Case 4.**  $r < 4s$ :

Since  $r < 4s$ , we have  $\mu(x_{s+3}) = 8s - r$ . Then  $\mu(x_{s+4}) = \max(\mu(x_{s+2}), 4s + \mu(x_{s+3})) = 12s - r$ , and by induction,  $\mu(x_i) = (i - s - 1)4s - r$  for  $i \geq s + 3$ . Therefore

$$\begin{aligned} \mu(x'_{2s+3}) &= \mu(x_{2s+3} - \frac{P_0(M^s)}{P_1(M^s)}(M^{2r} + \frac{P_2(M^s)}{P_0(M^s)}x_{2s+2})) \\ &= \max(\mu(x_{2s+3}), \max(2r - 4s + \mu(x_{2s+2}), -4s + \mu(x_{2s+2}))) \\ &= \max((s+2)4s - r, 2r - 4s + (s+1)4s - r, -4s + (s+1)4s - r) \\ &= \max(4s^2 + 4s - r, 4s^2 + r, 4s^2 - r), \end{aligned}$$

which is defined because the entries cannot be equal. Therefore,  $x'_{2s+3} \neq 0$ .



**Case 5.**  $r > 4s$ :

Here, we have  $\mu(x_{s+3}) = r$ . Then  $\mu(x_{s+4}) = \max(4s - r, 4s + r) = 4s + r$  since  $s > 0$ . Then by induction,  $\mu(x_i) = (i - s - 3)4s + r$  for  $i \geq s + 3$ . Therefore

$$\begin{aligned} \mu(x'_{2s+3}) &= \mu(x_{2s+3} - \frac{P_0(M^s)}{P_1(M^s)}(M^{2r} + \frac{P_2(M^s)}{P_0(M^s)}x_{2s+2})) \\ &= \max(\mu(x_{2s+3}), \max(2r - 4s + \mu(x_{2s+2}), -4s + \mu(x_{2s+2}))) \\ &= \max(4s^2 + r, 2r - 4s + (s - 1)4s + r, -4s + (s - 1)4s + r) \\ &= \max(4s^2 + r, 4s^2 - 8s - r, 4s^2 - 8s + r), \end{aligned}$$

which is defined because the entries cannot be equal. Therefore,  $x'_{2s+3} \neq 0$ .

We conclude that the matrix  $A$  has rank  $2s + 3$  for all  $s > 2$  and  $r$  relatively prime to  $s$ . Therefore, we have a single free variable, say  $Q_0$ , which determines an annihilator for  $T_n$ . Let  $Q(t, M, L)$  be the solution obtained this way such that its coefficients are relatively prime and in  $\mathbb{Z}[t^{\pm 1}, M]$  and let  $B(t, M) := \sum_{k=0}^{2s} c_k b(t, t^{2(s-1+k)}M^s)$ .

**Example 4.2.** We illustrate explicitly for the case  $s = 3$ . Setting  $Q_0 = 1$ , we have the matrix equation

$$\begin{bmatrix} P_0(t^4 M^3) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ P_1(t^4 M^3) & P_0(t^6 M^3) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ P_2(t^4 M^3) & P_1(t^6 M^3) & P_0(t^8 M^3) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & P_2(t^6 M^3) & P_1(t^8 M^3) & P_0(t^{10} M^3) & 0 & 0 & 0 & 0 & t^{-4r} M^{-r} \\ 0 & 0 & P_2(t^8 M^3) & P_1(t^{10} M^3) & P_0(t^{12} M^3) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & P_2(t^{10} M^3) & P_1(t^{12} M^3) & P_0(t^{14} M^3) & 0 & 0 & -t^{4r} M^r \\ 0 & 0 & 0 & 0 & P_2(t^{12} M^3) & P_1(t^{14} M^3) & P_0(t^{16} M^3) & t^{-6r} M^{-r} & 0 \\ 0 & 0 & 0 & 0 & 0 & P_2(t^{14} M^3) & P_1(t^{16} M^3) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & P_2(t^{16} M^3) & -t^{6r} M^r & 0 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ Q_2 \\ Q_1 \end{bmatrix} = \begin{bmatrix} -t^{-2r} M^{-r} \\ 0 \\ t^{2r} M^r \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying Cramer's rule gives us an annihilator of  $T_n$ :

$$Q(t, M, L) = 1 + \frac{\det A_9}{\det A} L + \frac{\det A_8}{\det A} L^2,$$

where  $A_i$  is the matrix  $A$  with the  $i$ -th column replaced by the vector on the right side of the equality. This solution exists since  $\det A \neq 0$ . We can get an annihilator with coefficients in  $\mathbb{Z}[t^{\pm 1}, M]$  by multiplying by a suitable element to clear the denominators.

It remains to check that  $Q(t, M, L)$  is an inhomogeneous recurrence relation for  $T_n$  rather than homogeneous. Suppose  $Q(t, M, L)T_n = 0$ . Then we have a homogeneous annihilator for  $J_{C,n}$  since

$$Q(t, M, L)(t^{2rs} M^{rs} L^2 - t^{-2rs} M^{-rs})J_{C,n} = 0.$$

It is proved in [8] that any recurrence polynomial of the colored Jones polynomial of a knot, when evaluated at  $t = -1$ , must contain the factor  $(L - 1)$ . However, here we have

$$Q(-1, M, L)(M^{rs} L^2 - M^{-rs}) = Q(-1, M, L)M^{rs}(L - M^{-rs})(L + M^{rs}),$$

and  $rs \neq 0$ , so we must have  $L - 1$  dividing  $Q(-1, M, L)$ . We shall see shortly, however, that  $Q(-1, M, L)$  is an irreducible polynomial of  $L$ -degree 2 over  $\mathbb{C}(M, L)$ , so we cannot have  $L - 1$  dividing  $Q(-1, M, L)$ . We conclude that  $Q(t, M, L)T_n = B(t, M)$  is an inhomogeneous recurrence relation.  $\diamond$

We would now like to check the  $AJ$ -conjecture. Recall that equation (1.1) gives

$$A_C(M, L) = (L - 1)(M^{2rs}L^2 - 1)Red(Res_\lambda(\frac{A_E(M^s, \lambda)}{\lambda - 1}, \lambda^s - L))$$

up to a factor of a power of  $M$ . With  $Q(t, M, L)$  given in Proposition 4.1, we have an annihilator

$$(L - 1)B(t, M)^{-1}Q(t, M, L)(t^{2rs}M^{rs}L^2 - t^{-2rs}M^{-rs})J_{C,n} = 0,$$

so to verify the  $AJ$ -conjecture, it is enough to show that  $Q(-1, M, L)$  is equal to the remaining factor  $Red(Res_\lambda(\frac{A_E(M^s, \lambda)}{\lambda - 1}, \lambda^s - L))$  up to a factor of an element in  $\mathbb{C}(M)$ . In fact, we shall see that  $Res_\lambda(\frac{A_E(M^s, \lambda)}{\lambda - 1}, \lambda^s - L)$  is irreducible for all positive values of  $s$ , so there are no repeated factors, so we can ignore the function  $Red$ . Connecting these two polynomials is the focus of the next section.

**4.2. The Resultant.** The goal of this section is to prove the following proposition.

**Proposition 4.3.** *Let  $r, s$  be relatively prime integers,  $s > 1$ , and let  $\tilde{\alpha}_E(t, M, L)$  be the inhomogeneous annihilator of  $J_{E,n}$  given in equation (1.2). Let  $Q(t, M, L)$  be the polynomial given by Proposition 4.1. Then*

$$Q(-1, M, L) = C(M)Res_\lambda(\tilde{\alpha}_E(-1, M^s, \lambda), \lambda^s - L)$$

for some  $C(M) \in \mathbb{C}(M)$ .

Our method of attack is to show that in a commutative setting, the analogous problem to what we solved in the previous section has a straightforward solution via the resultant. Our brute force approach then reduces to more or less computing the resultant when we evaluate at  $t = -1$ .

Let us recall the definition of the resultant.

**Definition 4.4.** *Let  $\mathbb{K}$  be a field and let  $f(x) = f_n x^n + \dots + f_0$  and  $g(x) = g_m x^m + \dots + g_0$  be polynomials in  $\mathbb{K}[x]$  of degree  $n$  and  $m$  respectively. Then the **resultant of  $f$  and  $g$**  is the determinant of the  $(m + n) \times (m + n)$  **Sylvester matrix of  $f$  and  $g$** ,*

$$Res(f, g) = \begin{vmatrix} f_0 & & & g_0 & & & \\ f_1 & f_0 & & g_1 & g_0 & & \\ f_2 & f_1 & \ddots & g_2 & g_1 & \ddots & \\ \vdots & f_2 & \ddots & f_0 & \vdots & g_2 & \ddots & g_0 \\ f_n & \vdots & \ddots & f_1 & g_m & \vdots & \ddots & g_1 \\ & f_n & \ddots & f_2 & & g_m & \ddots & g_2 \\ & & \ddots & \vdots & & & \ddots & \vdots \\ & & & f_n & & & & g_m \end{vmatrix}.$$

$\underbrace{\hspace{10em}}_{m \text{ columns}} \quad \underbrace{\hspace{10em}}_{n \text{ columns}}$

Moreover, given two polynomials over two variables,  $f(x, y) = f_n(x)y^n + \dots + f_0(x)$  and  $g(x, y) = g_m y^m + \dots + g_0(x)$ , we define the **resultant of  $f$  and  $g$  with respect to  $y$  or eliminating  $y$** , denoted  $Res_y(f, g)$ , to be  $Res(f, g)$  over the field  $\mathbb{K}(x)$ .

A key property of the resultant  $Res_y(f, g)(x)$ , as stated in [5, Ch. 12, p. 398], is that  $\alpha$  is a root of  $Res_y(f, g)$  if and only if either  $f(\alpha, y)$  and  $g(\alpha, y)$  have a common root or  $f_n(\alpha) = g_m(\alpha) = 0$ . Thus, given a system of two polynomial equations in two variables

$$\begin{cases} f(x, y) &= 0 \\ g(x, y) &= 0, \end{cases}$$

the resultant can be used to eliminate one of the variables from the system.

We now consider the commutative analog to the problem we solved in Section 4.1. That is, given a recurrence relation of a sequence, how can we find a recurrence relation of a certain related sequence?

Fix a field  $F := \mathbb{C}(M)$  and consider  $\mathcal{S} := \{S : \mathbb{N} \rightarrow F\}$ , the set of  $F$ -valued sequences. Then  $\mathcal{S}$  is an  $F[L]$ -module, where  $L \cdot S_n := S_{n+1}$  for all  $S \in \mathcal{S}$  and an element  $c \in F$  acts on a sequence by multiplication. Given a sequence  $S \in \mathcal{S}$ , we can define the set  $\mathcal{A}_S := \{P(L) \in F[L] \mid P(L) \cdot S_n = 0 \text{ for all } n\}$ , called the *annihilator ideal* of  $S$ . Moreover,  $F[L]$  is a principal ideal domain. A generator of the principal ideal  $\mathcal{A}_S$  is thus an element of  $\mathcal{A}_S$  of minimal degree and corresponds to a minimal order homogeneous recurrence relation of  $S$ .

Similarly, the set  $\hat{\mathcal{A}}_S := \{P(L) \in F[L] \mid \text{there exists } b \in F \text{ such that for every } n \in \mathbb{N}, P(L) \cdot S_n = b\}$  is a principal ideal, which consists of polynomials giving rise to inhomogeneous recurrence relations.

Suppose  $S \in \mathcal{S}$  satisfies a minimal recurrence relation  $\sum_{i=0}^d P_i S_{n+i} = b$ . Then  $S$  has an (inhomogeneous) annihilator  $P(L) = \sum_{i=0}^d P_i L^i$ . Consider the new sequence  $T_n := S_{kn}$  for some  $k > 1$ . How do we find a recurrence relation for  $T_n$ ?

Let  $\mathcal{S}_k := \{T \in \mathcal{S} \mid T_n = S_{kn+i} \text{ for some } S_n \in \mathcal{S}, i \in \mathbb{Z}\}$ . Now  $\mathcal{S}_k$  is a  $F[L, \lambda]$ -module, where  $L \cdot T_n = S_{k(n+1)+i} = S_{kn+k+i} = T_{n+1}$  and  $\lambda \cdot T_n = S_{k(n+1/k)+i} = S_{kn+i+1} \in \mathcal{S}_k$ . Then if  $T_n = S_{kn}$  and  $S_n$  is annihilated by  $P(L)$ , then  $\sum_{i=0}^d P_i S_{kn+i} = b$  as well, so we have  $T_n$  is annihilated by the polynomial  $P(\lambda)$ . We would like to obtain an annihilator in the variable  $L$  only, so we want to eliminate  $\lambda$ . We can do this using the fact that  $\lambda^k$  acts as  $L$ . Then we effectively want to solve the system of polynomial equations

$$\begin{cases} P(\lambda) &= 0 \\ \lambda^k - L &= 0. \end{cases}$$

Consider the polynomial  $R(L) = Res_\lambda(P(\lambda), \lambda^k - L)$ , the resultant of the two polynomials with respect to the variable  $\lambda$ . We check that  $R(L)$  is actually what we want, an annihilator of  $S_{kn}$ .

**Lemma 4.5.** *Let  $P(L)$  be a (possibly inhomogeneous) annihilator of the sequence  $S_n$  and suppose  $P(L)$  has no repeated roots in the algebraic closure  $\overline{F}$ . Then  $R(L) = Res_\lambda(P(\lambda), \lambda^k - L)$  is an annihilator of  $S_{kn}$ .*

*Proof.* Let  $\beta \in \overline{F}$  be a root of  $P(\lambda)$  and let  $\alpha := \beta^k$ . Then  $\beta$  is a common root of  $P(\lambda)$  and  $\lambda^k - \alpha$ . Then by the previously mentioned key property of resultants,  $\alpha$  is a root of  $R(L)$ , and thus  $\beta$  is a root of  $R(\lambda^k)$ . Therefore every root of  $P(\lambda)$  is also a root of  $R(\lambda^k)$ , so  $P(\lambda)$  divides  $R(\lambda^k)$  since  $P(\lambda)$  has no repeated roots. Since  $P(\lambda)$  annihilates  $S_{kn}$ , it follows that  $R(\lambda^k)$  is

also an annihilator of  $S_{kn}$ . Since  $\lambda^k$  has the same action as  $L$ , we have  $P(L)$  is an annihilator of  $S_{kn}$ , as needed.  $\diamond$

We will also require the following lemma.

**Lemma 4.6.** *Let  $k > 1$  and let  $P(\lambda) \in F[L, \lambda]$  be an irreducible polynomial of  $\lambda$ -degree  $d \leq 3$ . If every root  $\beta \in \overline{F}$  of  $P$  satisfies  $\beta^k \notin F$ , then  $R(L) = \text{Res}_\lambda(P(\lambda), \lambda^k - L)$  is an irreducible polynomial of  $L$ -degree  $d$ .*

*Proof.* Suppose  $\alpha \in \overline{F}$  is a root of  $R(L)$ . Then there is  $\beta \in \overline{F}$  such that  $\beta^k = \alpha$  and  $P(\beta) = 0$ , and so  $\alpha \notin F$  by assumption. Therefore,  $R(L)$  has no roots in  $F$ , so it has no linear factors, and since  $R(L)$  is also of degree  $d \leq 3$  by the definition of the resultant, we conclude that  $R(L)$  is irreducible.  $\diamond$

We are now ready to connect the polynomial  $Q(t, M, L)$  of Proposition 4.1 to the  $A$ -polynomial of  $E$ .

*Proof of Proposition 4.3.* We know that the polynomial  $R(L) := \text{Res}_\lambda(\tilde{\alpha}_E(-1, M^s, \lambda), \lambda^s - L)$  solves the problem of finding an annihilator for the sequence  $S_{sn}$  if  $S_n$  has the annihilator  $\tilde{\alpha}_E(-1, M^s, L)$ . We will construct a sequence  $S_n$  for which both  $Q(-1, M, L)$  and this resultant are annihilators, and since  $\hat{\mathcal{A}}_S$  is a principal ideal, the resultant is irreducible, and the polynomials have the same degree, they will be the same up to a unit.

Fix the field  $\mathbb{C}(M)$ . Since  $\tilde{\alpha}_E(-1, M^s, L)$  has  $L$ -degree 2, it gives rise to an inhomogeneous recurrence relation

$$P_2(-1, M^s)S_{n+2} + P_1(-1, M^s)S_{n+1} + P_0(-1, M^s)S_n = b(-1, M^s)$$

for some sequence  $S_n$ , and fixing two initial conditions defines  $S_n$ , so let  $S_0 = 1$  and  $S_1 = 1$ .

Let  $T_n = M^r S_{s(n+1)+1} - M^{-r} S_{s(n+1)-1}$  for some  $r \in \mathbb{Z}$ . By Lemma 4.5, we know that  $R(L)S_{sn} = B(M)$  for some  $B(M) \in \mathbb{C}(M)$ . Moreover, we have

$$\begin{aligned} R(L) \cdot T_n &= M^r R(L) \cdot S_{s(n+1)+1} - M^{-r} R(L) \cdot S_{s(n+1)-1} \\ &= M^r R(L) \lambda^{s+1} \cdot S_{sn} - M^{-r} R(L) \lambda^{s-1} \cdot S_{sn} \\ &= (M^r - M^{-r})B(M), \end{aligned}$$

so  $R(L)$  is an annihilator of  $T_n$ .

We claim  $T_n$  is not a constant sequence. We show this by computing  $\mu(T_n)$ , the  $M$ -degree of  $T_n$ , and showing it is not zero. First, we have

$$S_{n+2} = -\frac{P_1(-1, M^s)}{P_2(-1, M^s)}S_{n+1} - \frac{P_0(-1, M^s)}{P_2(-1, M^s)}S_n + \frac{b(-1, M^s)}{P_2(-1, M^s)},$$

and  $\mu(P_0(-1, M^s)) = \mu(P_2(-1, M^s)) = 8s$ ,  $\mu(P_1(-1, M^s)) = 12s$ , and  $\mu(b(-1, M^s)) = 11s$ , so

$$\mu(S_{n+2}) = \max(4s + \mu(S_{n+1}), \mu(S_n), 11s), \quad n \geq 0$$

while  $\mu(S_0) = \mu(S_1) = 0$ , and so  $\mu(S_2) = 11s$  and  $\mu(S_n) = 11s + (n-2)4s = 4sn + 3s$  for  $n \geq 3$ . Thus

$$\begin{aligned}\mu(T_n) &= \max(r + 4s(s(n+1) + 1) + 3s, -r + 4s(s(n+1) - 1) + 3s) \\ &= \max(r + 4s^2n + 4s^2 + 7s, -r + 4s^2n + 4s^2 - s)\end{aligned}$$

since  $r + 4s^2n + 4s^2 + 7s = -r + 4s^2n + 4s^2 - s$  if and only if  $r = -4s$ , but  $r$  and  $s$  are relatively prime. Since  $\mu(T_n) \neq 0$  and is finite,  $T_n$  is not a constant sequence. In particular,  $\hat{\mathcal{A}}_T \neq \mathbb{C}(M)$ .

Next, we claim that  $R(L)$  is irreducible over  $\mathbb{C}(M)$  and consequently a generator of  $\hat{\mathcal{A}}_T$ . By Lemma 4.6, it is enough to show that for every root  $\beta \in \overline{\mathbb{C}(M)}$  of  $\tilde{\alpha}_E(-1, M^s, \lambda)$ ,  $\beta^k \notin \mathbb{C}(M)$ .

Suppose  $P_2(-1, M^s)\beta^2 + P_1(-1, M^s)\beta + P_0(-1, M^s) = 0$ . Since  $\tilde{\alpha}_E(-1, M^s, \lambda)$  is irreducible over  $\mathbb{C}(M)$ , we know  $\beta \notin \mathbb{C}(M)$ . Then

$$\beta^2 = -\frac{P_1(-1, M^s)}{P_2(-1, M^s)}\beta - \frac{P_0(-1, M^s)}{P_2(-1, M^s)},$$

which is not in  $\mathbb{C}(M)$  since  $P_1(-1, M^s) \neq 0$ . We claim that for all  $k \in \mathbb{N}$ ,  $\beta^k = a_k\beta + b_k$  for some  $a_k, b_k \in \mathbb{C}(M)$ ,  $a_k \neq 0$ . We show this by computing  $\mu(a_k)$ . We have  $\mu(a_1) = \mu(1) = 0$ ,  $\mu(a_2) = \mu(P_1(-1, M^s)) - \mu(P_2(-1, M^s)) = 4s$ ,  $\mu(b_1) = \mu(0) = -\infty$ ,  $\mu(b_2) = 0$ . Let  $k > 2$ , and assume that for all  $i < k$ ,  $\mu(a_i) = 4s(i-1)$ . Then we have

$$\begin{aligned}\beta^k &= (a_{k-1}\beta + b_{k-1})\beta \\ &= a_{k-1}\beta^2 + b_{k-1}\beta \\ &= a_{k-1}\left(-\frac{P_1(-1, M^s)}{P_2(-1, M^s)}\beta - \frac{P_0(-1, M^s)}{P_2(-1, M^s)}\right) + b_{k-1}\beta \\ &= \left(-\frac{P_1(-1, M^s)}{P_2(-1, M^s)}a_{k-1} + b_{k-1}\right)\beta - \frac{P_0(-1, M^s)}{P_2(-1, M^s)}a_{k-1},\end{aligned}$$

which means  $b_k = -\frac{P_0(-1, M^s)}{P_2(-1, M^s)}a_{k-1}$ , so  $\mu(b_k) = \mu(a_{k-1})$ , hence  $\mu(b_{k-1}) = \mu(a_{k-2})$ . Thus

$$\begin{aligned}\mu(a_k) &= \mu\left(-\frac{P_1(-1, M^s)}{P_2(-1, M^s)}a_{k-1} + b_{k-1}\right) \\ &= \max(4s + \mu(a_{k-1}), \mu(b_{k-1})) \\ &= \max(4s + 4s(k-2), 4s(k-3)) \\ &= 4sk - 4s,\end{aligned}$$

so by induction,  $\mu(a_k) > 0$  for all  $k$  and hence  $\beta^k$  is never in  $\mathbb{C}(M)$ . We conclude that  $R(L)$  is irreducible.

Now, consider this alternate solution to the same problem, in the manner of the proof of Proposition 4.1. We know  $\tilde{\alpha}_E(-1, M^s, \lambda) \cdot S_{sn} = b(-1, M^s)$ , so we have the equations

$$\begin{aligned}\tilde{\alpha}_E(-1, M^s, \lambda) \cdot S_{s(n+1)-1+j} &= \sum_{i=0}^2 P_i(-1, M^s) S_{s(n+1)-1+i+j} \\ &= b(-1, M^s)\end{aligned}$$

for any  $j$ . We suspect we can find an inhomogeneous annihilator  $\hat{Q}_2 L^2 + \hat{Q}_1 L + \hat{Q}_0$  of  $T_n$  by finding a suitable linear combination of these equations:

$$\sum_{j=0}^{2s} c_j \sum_{i=0}^2 P_i(-1, M^s) S_{s(n+1)-1+i+j} = \sum_{i=0}^2 \hat{Q}_i T_{n+i}.$$

Setting the coefficients of each  $S_{sn+k}$  equal to zero, we get a linear system in  $2s + 3$  equations and  $2s + 4$  unknowns. The resulting matrix of coefficients is exactly the matrix  $A$  in equation 4.3 after setting  $t = -1$ . By the same argument as before, we see that we can find nonzero  $\hat{Q}_0$ ,  $\hat{Q}_1$ , and  $\hat{Q}_2$ , so  $\hat{Q}(L) = \hat{Q}_2 L^2 + \hat{Q}_1 L + \hat{Q}_0$  is an inhomogeneous annihilator of  $T_n$  of degree 2, and this  $\hat{Q}(L)$  is exactly  $Q(-1, M, L)$  up to a factor of a rational function in  $\mathbb{C}(M)$ .

Then since  $R(L)$  is a generator of  $\hat{\mathcal{A}}_T$ ,  $R(L)$  divides  $Q(-1, M, L)$ , and since they are both of degree 2, we have  $C(M)R(L) = Q(-1, M, L)$  for some  $C(M) \in \mathbb{C}(M)$ , which completes the proof.

◇

Finally, since  $E$  satisfies the  $AJ$ -conjecture, we know  $\tilde{\alpha}_E(-1, M, L) = f(M) \frac{A_E(M, L)}{L-1}$  for some polynomial  $f(M)$ . The resultant being the determinant of the Sylvester matrix gives us

$$\text{Res}_\lambda(\tilde{\alpha}_E(-1, M^s, \lambda), \lambda^s - L) = \text{Res}_\lambda(f(M^s) \frac{A_E(M^s, \lambda)}{\lambda - 1}, \lambda^s - L) = f(M^s)^s \text{Res}_\lambda(\frac{A_E(M^s, \lambda)}{\lambda - 1}, \lambda^s - L).$$

It follows that  $Q(-1, M, L)$  is equal to  $\text{Res}_\lambda(\frac{A_E(M^s, \lambda)}{\lambda - 1}, \lambda^s - L)$  up to a factor of an element in  $\mathbb{C}(M)$ , and consequently the  $AJ$ -conjecture is satisfied for the knot  $C$ .

**4.3. Minimal degree of the recurrence relation.** Suppose there is a recurrence relation of  $L$ -degree at most 4 of  $J_{C,n}$

$$D_4 J_{C,n+4} + D_3 J_{C,n+3} + D_2 J_{C,n+2} + D_1 J_{C,n+1} + D_0 J_{C,n} = 0.$$

Combining equation (3.1) with the definition of  $T_n$  in equation (4.1), we have

$$J_{C,n+2} = t^{-4rs} M^{-2rs} J_{C,n} + t^{-2rs} M^{-rs} T_n,$$

so we can use this to simplify our recurrence:

$$\begin{aligned} 0 &= D_4 J_{C,n+4} + D_3 J_{C,n+3} + D_2 J_{C,n+2} + D_1 J_{C,n+1} + D_0 J_{C,n} \\ &= D_4 (t^{-12rs} M^{-2rs} J_{C,n+2} + t^{-6rs} M^{-rs} T_{n+2}) + D_3 (t^{-8rs} M^{-2rs} J_{C,n+1} + t^{-4rs} M^{-rs} T_{n+1}) \\ &\quad + D_2 (t^{-4rs} M^{-2rs} J_{C,n} + t^{-2rs} M^{-rs} T_n) + D_1 J_{C,n+1} + D_0 J_{C,n} \\ &= (D_4 t^{-16rs} M^{-4rs} + D_2 t^{-4rs} M^{-2rs} + D_0) J_{C,n} + (D_3 t^{-8rs} M^{-2rs} + D_1) J_{C,n+1} \\ &\quad + (D_4 t^{-6rs} M^{-rs}) T_{n+2} + (D_3 t^{-4rs} M^{-rs}) T_{n+1} + (D_4 t^{-14rs} M^{-3rs} + D_2 t^{-2rs} M^{-rs}) T_n, \end{aligned}$$

and by Proposition 4.1, we have

$$T_{n+2} = \frac{B}{Q_2} - \frac{Q_1}{Q_2} T_{n+1} - \frac{Q_0}{Q_2} T_n,$$

so making this substitution yields

$$\begin{aligned} 0 &= (D_4 t^{-16rs} M^{-4rs} + D_2 t^{-4rs} M^{-2rs} + D_0) J_{C,n} + (D_3 t^{-8rs} M^{-2rs} + D_1) J_{C,n+1} \\ &\quad + (D_4 (t^{-14rs} M^{-3rs} - \frac{Q_0}{Q_2} t^{-6rs} M^{-rs}) + D_2 t^{-2rs} M^{-rs}) T_n \\ &\quad + (D_3 t^{-4rs} M^{-rs} - D_4 \frac{Q_1}{Q_2} t^{-6rs} M^{-rs}) T_{n+1} + D_4 \frac{B}{Q_2} t^{-6rs} M^{rs}, \end{aligned}$$

and multiplying both sides of the equation by  $Q_2$  gives us something of the form

$$0 = D'_4 J_{C,n} + D'_3 J_{C,n+1} + D'_2 T_n + D'_1 T_{n+1} + D'_0$$

where the  $D'_i$  are Laurent polynomials in  $t$  and  $M$ . It is easy to see that  $D'_4 = \dots = D'_0 = 0$  implies  $D_4 = \dots = D_0 = 0$ . Notice that if the degree of the recurrence polynomial is less than 4, then  $D_4 = 0$  and so  $D'_0 = 0$ .

**Lemma 4.7.** *When  $r > 4s$  or  $r < -4s$ , if  $D'_0 = 0$ , then  $D'_i = 0$  for  $i = 1, \dots, 4$  as well.*

*Proof.* Suppose  $r > 4s$  and  $D'_4 \neq 0$ . As in the  $s = 2$  case, we compare the lowest degrees in  $t$  of the summands. We need another  $D'_i$  to be nonzero in order to cancel  $D'_4 J_{C,n}$ , so we examine the cases.

Notice that  $\ell[T_n] = \min(2r(n+1) + \ell[J_{E,s(n+1)+1}], -2r(n+1) + \ell[J_{E,s(n+1)-1}])$ , so by Lemma 2.1,

$$\ell[T_n] = -2r(n+1) - 4s^2 n^2 + (10s - 8s^2)n - 4s^2 + 10s - 4.$$

We know  $\ell[J_{C,n}]$  has a coefficient of  $-rs$  on  $n^2$  by Lemma 2.2, and  $-rs < -4s^2$ , so the lowest degree  $D'_4 J_{C,n}$  cannot be canceled by the lowest degree in  $D'_2 T_n$  or  $D'_1 T_{n+1}$ , since  $\ell[D'_i]$  is only linear in  $n$  for sufficiently large  $n$ . Then we must have  $D'_3 \neq 0$ . But this gives us

$$\begin{aligned} \ell[D'_4] - \ell[D'_3] &= \ell[J_{C,n+1}] - \ell[J_{C,n}] \\ &= -2rsn - rs + (-1)^n (s-2)(4s-r-2), \end{aligned}$$

which is eventually linear on the left but alternating on the right, which is impossible. Therefore,  $D'_4 = 0$ . Similarly,  $D'_3 = 0$ , and when  $r < -4s$ , using the highest degree in  $t$  gives  $D'_4 = D'_3 = 0$  as well.

Next, given that  $D'_4 = D'_3 = 0$ , we have

$$0 = D'_2 T_n + D'_1 T_{n+1}.$$

Suppose  $D'_2 \neq 0$  and  $D'_1 \neq 0$ . Then we have a first order homogeneous recurrence relation for  $T_n$ . We find a contradiction using Lemma 2.3. It is not hard to see that the breadth  $h[T_n] - \ell[T_n]$  is quadratic in  $n$ . Also, notice that

$$\begin{aligned} T_{-n} &= t^{2r(-n+1)} J_{E,s(-n+1)+1} - t^{-2r(-n+1)} J_{E,s(-n+1)-1} \\ &= -t^{-2r(n-1)} J_{E,s(n-1)-1} + t^{2r(n-1)} J_{E,s(n-1)+1} \\ &= T_{n-2}, \end{aligned}$$

so by Lemma 2.3, any homogeneous recurrence relation of  $T_n$  has order at least 2, which is a contradiction. Therefore one of  $D'_2$  or  $D'_1$  is zero, but since  $T_n$  is not zero, we must have  $D'_1 = D'_2 = 0$ , which completes the proof.  $\diamond$

This implies that the recurrence polynomial of  $C$  has L-degree 4.

**Lemma 4.8.** *When  $r > 4s$  or  $r < -4s$ , we have  $D'_i = 0$  for  $i = 0, \dots, 4$ .*

*Proof.* Noting that  $\ell[D'_0]$  is linear in  $n$ , the proof that  $D'_4 = D'_3 = 0$  is the same.

Since  $D'_4 = D'_3 = 0$ , we have

$$0 = D'_2 T_n + D'_1 T_{n+1} + D'_0.$$

This is a degree 1 inhomogeneous recurrence relation for  $T_n$ . The rest of the proof is analagous to the proof of Lemma 3.3 in the  $s = 2$  case.  $\diamond$

We conclude that

$$(L - 1)B(t, M)^{-1}Q(t, M, L)(t^{2rs}M^{rs}L^2 - t^{-2rs}M^{-rs})$$

is the recurrence polynomial of the  $(r, s)$ -cabled knot  $C$  over the figure eight knot if  $s > 2$  and  $r > 4s$  or  $r < -4s$  up to a factor of an element in  $\mathbb{C}(t, M)$ .

## REFERENCES

- [1] **D. Bar-Natan, S. Garoufalidis**, *On the Melvin-Morton-Rozansky conjecture*, Invent. Math. 125 (1996), 1031-1033.
- [2] **D. Cooper, M. Culler, H. Gillet, D. Long, P. Shalen**, *Plane curves associated to character varieties of 3-manifolds*, Invent. Math. 118 (1994) 47-84.
- [3] **S. Garoufalidis**, *On the characteristic and deformation varieties of a knot*, Proceedings of the Casson Fest, Geometry and Topology Monographs 7 (2004) 291-309.
- [4] **S. Garoufalidis, T. Le**, *The colored Jones function is  $q$ -holonomic*, Geometric Topology 9 (2005) 1253-1293.
- [5] **I. Gelfand, M. Kapranov, A. Zelev**, *Discriminants, Resultants, and Multidimensional Determinants*, Birkhäuser, Boston, 1994.
- [6] **K. Hikami**, *Difference equation of the colored Jones polynomial for torus knot*, Internat. J. Math. 15 (2004) 959-965.
- [7] **R. Kashaev**, *The hyperbolic volume of knots from the quantum dilogarithm*, Lett. Math. Phys. 39 (1997), no. 3, 269-275.
- [8] **T. Le**, *The colored Jones polynomial and the  $A$ -polynomial of knots*, Advances in Math. 207 (2006) 782-804.
- [9] **T. Le, A. Tran**, *On the AJ-conjecture for knots*, preprint, math.GT/1111.5258.
- [10] **H. Morton**, *The coloured Jones function and Alexander polynomial for torus knots*, Proc. Cambridge Philos. Soc. 117 (1995) 129-135.
- [11] **H. Murakami, J. Murakami**, *The colored Jones polynomials and the simplicial volume of a knot*, Acta. Math. 186 (2001) 85-104.
- [12] **Y. Ni, X. Zhang**, *Detection of knots and a cabling formula for  $A$ -polynomials*, preprint, math.GT/1411.0353.
- [13] **D. Ruppe, X. Zhang**, *The AJ-conjecture and cabled knots over torus knots*, preprint, math.GT/1403.1858.
- [14] **T. Takata**, *The colored Jones polynomial and the  $A$ -polynomial for twist knots*, preprint, math.GT/0401068.
- [15] **A. Tran**, *Proof of a stronger version of the AJ conjecture for torus knots*, Algebraic and Geometric Topology 13 (2013), no. 1, 609-624.
- [16] **A. Tran**, *On the AJ conjecture for cables of the figure eight knot*, New York J. Math. 20 (2014), 727-741.
- [17] **R. van der Veen**, *A cabling formula for the colored Jones polynomial*, Oberwolfach Reports: Low-Dimensional Topology and Number Theory (2010) 2101-2163.

DEPARTMENT OF MATHEMATICS, UNIVERSITY AT BUFFALO, BUFFALO, NY, 14214-3093, USA.

*E-mail address:* dennisru@buffalo.edu